JULES HENRI POINCARÉ.

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In a much quoted sentence, Klein has said "We shall have a picture of the development of mathematics if we imagine a chain of lofty mountains as representative of the men of the eighteenth century, terminating in a mighty outlying summit,—Gauss, and then a broader, hilly country of lower elevation; but teeming with new elements of life." This was written in 1893 and would perhaps still be received as the truth by most observers. During the ensuing period it has, however, become more and more evident that two of the contemporary hills rise quite above the others, and it may be that when they are seen in perspective they will compare favorably with the more distant mountains.

Hilbert and Poincaré are associated and contrasted in the minds of all students of mathematics both by the brilliance of their achievements and the difference of their methods. To enter into a comparative study of these two great men would not be an appropriate exercise for this occasion, but to have suggested it may help to indicate the relation of Poincaré to the time in which he lived.

Aside from his intellectual triumphs, which one could adequately comprehend only by reading a series of his papers, the life of Poincaré presents little of interest. He was born at Nancy on the 29th of April, 1854. His unusual gifts were recognized early, so that he had an excellent education. He received the degrees of Bachelor of Letters and of Science in 1871 and that of Mining Engineer in 1879. He was attached in one capacity or another to the Department of Mines of the French government for the rest of his life, but not in such a way as to interfere with his scientific work.

In the year 1879 he also received his doctorate of Science from the University of Paris. He was immediately made a member of the faculty of science at Caen. From there he was called to the University of Paris in 1881. In 1886 he was appointed professor

of physics and of the calculus of probabilities in the University of Paris and in 1896 became professor of mathematical astronomy at the same university. In 1904 he was also made professor of general astronomy at the École Polytechnique. Since 1902 he occupied the chair of electricity at the École professionnelle supérieure des Postes et des Télégraphes. He died on the 17th of July, 1912.

The importance of his scientific contributions was recognized from the very beginning of his career. He received practically all the distinctions which are open to a mathematician. Among the most notable were: Election to the Academy of Science of France (Section of Geometry) in 1887, election to the French Academy in 1908, and the Bolyai Prize for excellence in all fields of mathematics in 1905. He was elected a member of the American Philosophical Society in 1899.

Poincaré was particularly distinguished among his contemporaries by the wide range of his creative power. He left behind enduring works not only in the several branches of pure mathematics but in astronomy, physics and philosophy. He has often been described as the last of the universals. Indeed in this respect as well as in the brilliance of his individual works, he is like those earlier heroes of science whom Klein compared to the chain of lofty mountains.

It goes without saying that one could not expect to give an adequate account of Poincaré's complete work in a short address like this one. I shall try, however, to mention certain main divisions of his work, taking them up in an order which is roughly chronological. Naturally, the periods to which I shall refer all overlap but I shall try to arrange them according to the dates of the central papers in each subject.

Poincaré's doctoral dissertation, which was his first published work of importance, appeared in 1879. Its title was "On the Properties of Functions Defined by Partial Differential Equations" and it supplies the existence theorem for solutions in the neighborhood of singular points of a very general type. This memoir initiated a long series of brilliant contributions to the theory of differential equations, especially to that of linear differential equations. Most of these papers appeared in the period before 1886.

These studies led directly to his discoveries in the field of automorphic functions where Poincaré achieved his first great celebrity. Like most first-class things in modern mathematics, it is impossible to describe these functions briefly in a non-technical discussion.[1] We must be content to characterize them as the nearest lying and most beautiful generalizations of the trigonometric and elliptic functions. Poincaré deserves to be classed as one of the founders of this branch of mathematics, for to him are due some of the main outlines of the theory and the main existence theorems. Poincaré's last important contribution was his memoir on the zeta-fuchsian functions in the *Acta Mathematica* for 1884. Since then this work has been carried forward chiefly by Klein and his students.

This series of contributions to function theory was followed in 1885 by his epoch-making memoir on the figure of equilibrium of a rotating fluid mass. In this work he not only solved the problem of stability for the previously known figures of equilibrium, the ellipsoids of Maclaurin and of Jacobi, but he also discovered a whole class of new figures of equilibrium. This work is important not only on account of the particular new figures (the pear-shaped figures) which it put in evidence, but also on account of its method. I need mention only the theorem on the exchange of equilibrium.

In 1890 he made a still greater contribution to mathematical astronomy in his memoir "On the Problem of Three Bodies and the Equations of Dynamics." Here he brought into existence a general theory of periodic orbits and disproved the existence of further new integrals which are analytic functions of the masses. These matters and many others, such as the integral invariants and asymptotic solutions, are the subject of his three volumes (1892, 1893 and 1899) on "Les Méthodes Nouvelles de la Méchanique Céleste." Later on he published three more volumes entitled "Leçons de Méchanique Céleste" in which he developed some of the classical theories from new points of view.

The problems of celestial mechanics continued to occupy the mind of Poincaré till the end of his life. In his last paper, which

[1] This difficulty must indeed be my excuse for the summary way in which I shall have to refer to the rest of Poincaré's work.

appeared in print shortly after his death, he shows how to reduce one of the problems regarding the existence of periodic orbits to a geometric problem, which, however, he was unable to solve. He apologizes for putting forth such an incomplete result on the ground that at his age (he was only 58) he could not feel confident of returning to the problem in the future and solving it completely. One cannot avoid the impression that he felt that his career was very nearly at an end. It will doubtless interest this audience to know that a proof of Poincaré's theorem has already been found by a young American mathematician, Professor G. D. Birkhoff, of Harvard.

We cannot here dwell longer on the astronomical work of Poincaré. We must pass over without particular mention his work on the figure of the earth, on the tides, and on the lunar theory, as well as his recent book on *cosmogony*.

Poincaré is the author of at least fourteen advanced text-books in various branches of physics. Among the titles we find Capillarity, Elasticity, Vortices, Heat, Thermodynamics, Optics, Electricity, Wireless Telegraphy, etc. These are chiefly reproductions of his courses of lectures at the Sorbonne. He also wrote a large number of papers and memoirs on physical topics, especially on Hertzian waves and on the theory of electrons. On the whole, however, his work in physics cannot be compared in importance with his fundamental contributions to mathematics and astronomy. His work on the differential equations of physics and on Dirichlet's principle, which one might be expected to mention here, is of more consequence to mathematics than to physics.

We must now turn to another main division of his work in a domain of pure mathematics. Already in 1883 he had published an important paper (in the *Acta Mathematica*) laying the foundations of the theory of functions of two complex variables. In 1887 he published his memoir on the residues of double integrals, which furnished one of the chief tools for the theory of algebraic functions of two variables, a theory which has since been built up chiefly by his colleague Picard, by Poincaré himself, and by many brilliant Italian geometers.

The theory of algebraic functions of one variable has as its most striking auxiliary the manifolds of two dimensions known as Riemann surfaces, and the theory of the connectivity of Riemann surfaces is the main object of the analysis situs of two dimensions. A generalization of this theory to manifolds of any number of dimensions was foreseen to some extent by Riemann himself and to a larger degree by Betti, who discovered a set of invariants of n-dimensional manifolds which are known as the Betti numbers. Little real progress, however, had been made till Poincaré took the question up, modified the Betti definitions, showed how the modified Betti numbers satisfy a generalization of Euler's theorem for polyhedra, and introduced an entirely new set of constants, the coefficients of torsion. This work is contained in a series of memoirs of which the first appeared in 1892 and the last in 1904. They were accompanied and followed by a number of papers in which analysis situs is applied to the theory of algebraic functions of two and more variables.

I have now mentioned what appear to me the most important achievements of Poincaré, grouping them together in four classes which, as I said, correspond very roughly to a chronological order, The four sections of his work which I have signalized are his contributions (1) to the classical theory of differential equations and the theory of automorphic functions, (2) to the theory of stability and of the differential equations of celestial mechanics, (3) to physics and (4) to analysis situs and the theory of functions of several variables.

Another side of Poincaré's intellectual activity which has attracted more general attention than any of his capital achievements in pure science is represented by his semi-philosophical books, "Science and Hypothesis," "The Value of Science," and "Science and Method." These books have been translated into most of the modern languages, including English, and have received much attention and praise. They are characterized by a clarity and hard-headed "common sense" which is more often sought than found in this class of literature.

After noticing the works for which Poincaré was chiefly famous, there still remains a host of mathematical papers which would be

sufficient to rank him ahead of most of his contemporaries. One thinks first perhaps of his two papers on the uniformization of a general analytic function which appeared in 1883 and 1908 respectively. Then there are his papers on transcendental entire functions and on analytic functions which have lacunary spaces. He made several contributions to the theory of Abelian functions, the reduction of Abelian integrals, the theory of the zeros of theta functions. His paper on linear equations of finite differences has stimulated a great activity of research in that field. In *Liouville's Journal* for 1890 he investigated sets of functions satisfying what he called a theorem of multiplication, including particularly a new class of functions which he named after Cremona. He also deserves credit for establishing the convergence of Hill's infinite determinant and wrote several papers on integral equations and their applications. Also several papers on continuous groups, on hypercomplex numbers, on number theory, and on the relation of automorphic functions to number theory.[2]

The mathematical style of Poincaré was intensely modern. There are few purely formal theorems to his credit. Few of his results depend on long or difficult computations. He said of himself with a furtive touch of humor—the remark came in a paper relating to his and Darwin's work on the pear-shaped figures—that he was poor at arithmetic. He was good, on the other hand, at divining a general principle after seeing the least possible number of special cases. He was tremendously powerful at the essentially modern game of finding out all about a function irrespective of whether it could be adequately described by formulas of the classic type. In any problem he felt instinctively for the fundamental group and for the invariants thereof. He had an almost visual grasp of the properties of a figure of any number of dimensions which remain invariant under a continuous deformation, and this either in case of the small deformations that are considered in problems of stability or in the larger ones that constitute the subject matter of analysis situs. All this was combined with a sound judgment which always directed his

[2] A satisfactory bibliography of Poincaré's publications is to be found in "Savants du Jour, Henri Poincaré," by E. Lebon (Paris, 1912).

energies towards problems of which the importance cannot be contested. Nearly everything that he did has been the starting point for the researches of a considerable number of other scientists.

The chief criticism that has been directed against Poincaré is that he never actually completed his work in any one branch of study. That after having discovered a few of the fundamental theorems, his curiosity was satisfied and he was ready to swing to another branch, again to pick the choicest fruit and leave the less exhilarating tasks to his slower contemporaries and to the future. There is truth in the charge. He could never have done what he did in any other way. But what critic would not be glad to do the same thing if he could? Indeed, it seems to me that both the blame and the praise which Poincaré deserves are condensed in the epigram of Borel: "He was a conqueror, not a colonist."

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